15/1/2019

π.χ. Να υπολογιστεί η ορίζουσα:

$$\det\begin{pmatrix} x & 1 & \cdots & 1 \\ 1 & x & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & x \end{pmatrix} = \overset{\text{τελικά}}{(x+n-1)/(x-1)^{n-1}}$$

$$\det\begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = \det\begin{pmatrix} x+2 & 1 & 1 \\ x+2 & x & 1 \\ x+2 & 1 & x \end{pmatrix}$$

$\cancel{(x+2)(x-1)^2}$

$$(x+2)\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = (x+2)\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix}$$

$$(x+2)(x-1)^2$$

π.χ. Να υπολογιστεί η ορίζουσα:

$$\det\begin{pmatrix} 1 & x_1(x_1-1) & x_1^2(x_1-1) & \cdots & x_1^{n-1}(x_1-1) \\ 1 & x_2(x_2-1) & x_2^2(x_2-1) & \cdots & x_2^{n-1}(x_2-1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n(x_n-1) & x_n^2(x_n-1) & \cdots & x_n^{n-1}(x_n-1) \end{pmatrix}$$

Vandermore

θέλω ορίζουσα $n \times n$, άρα

$$\begin{pmatrix} 1 & x_1 & \dots & x_{n-1} \\ 1 & x_2^2 & \dots & x_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ 1 & x_n^n & \dots & x_{n-1}^n \end{pmatrix} = \prod_{n-1 \geq i \geq j \geq 1} (x_i - x_j)$$

$$\begin{pmatrix} 1 = x_1 - (x_1 - 1) & x_1^2(x_1-1) & \dots & x_1^{n-1}(x_1-1) \\ 1 = x_2 - (x_2-1) & x_2^2(x_2-1) & \dots & x_2^{n-1}(x_2-1) \\ \vdots & \vdots & & \vdots \\ 1 = x_n - (x_n - 1) & x_n^2(x_n-1) & \dots & x_n^{n-1}(x_n-1) \end{pmatrix} =$$

$$= (x_1-1)(x_2-1)\dots(x_n-1) \det \begin{pmatrix} 1 & x_1^2 & \dots & x_n^{n-1} \\ 1 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & x_1 - 1 & x_1(x_1-1) & \dots & x_1^{n-2}(x_1-1) \\ 1 & x_2 - 1 & x_2(x_2-1) & \dots & x_2^{n-2}(x_2-1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n - 1 & x_n(x_n-1) & \dots & x_n^{n-2}(x_n-1) \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & x_1 & x_1(x_1-1) & \dots & x_1^{n-2}(x_1-1) \\ 1 & x_2 & x_2(x_2-1) & \dots & x_2^{n-2}(x_2-1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n(x_n-1) & \dots & x_n^{n-2}(x_n-1) \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^2(x_1-1) & \dots, & x_1^{n-2}(x_1-1) \\ 1 & x_2 & x_2^2 & x_2^2(x_2-1) & \dots, & x_2^{n-2}(x_2-1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^2(x_n-1) & \dots & x_n^{n-2}(x_n-1) \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & x_1 & x_1^2 & \dots, & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots, & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

Για την (1) $= x_1 \dots x_n \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} -$

$$- (x_1-1)(x_2-1) \dots (x_n-1) \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} =$$

$$= (x_1 x_2 \dots x_n) - (x_1-1)(x_2-1) \dots (x_n-1) \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix}$$

$$= \left[\prod_{i=1}^{n} x_i - \prod_{i=1} (x_i - 1) \right] \prod_{n \geq i \geq j \geq 1} (x_i - x_j)$$

π.χ. Να βρεθεί η γραμμική απεικόνιση. $T:\mathbb{R}^3 \to \mathbb{R}$ με πίνακα.

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{pmatrix}$$ ως προς τις βάσεις

$\{(0,1,0),(1,0,1),(1,1,0)\}$ και $\{(1,0),(0,1)\}$

Επίσης να βρεθεί ο πυρήνας της και ένα συμπλήρωμά του.

$T(x,y,z)=\dots$

$$(x,y,z)=x(1,0,0)+y(0,1,0)+z(0,0,1)$$
$$T(0,1,1)=1(1,0)+2(0,1)=(1,2)$$
$$T(1,0,1)=2(1,0)+1(0,1)=(2,1)$$
$$T(1,1,0)=3(1,0)+7(0,1)=(3,7)$$

$$(x,y,z)=\alpha(0,1,1)+\beta(1,0,1)+\gamma(1,1,0) \Leftrightarrow$$
$$\Leftrightarrow (x,y,z)=(\beta+\gamma,\alpha+\gamma,\alpha+\beta)$$
$$\Leftrightarrow \left.\begin{aligned} x&=\beta+\gamma \\ y&=\alpha+\gamma \\ z&=\alpha+\beta \end{aligned}\right\} \Rightarrow y-x=\alpha-\beta \Rightarrow \alpha=y-x+\beta$$

$$z=\alpha+\beta \Rightarrow z=y-x+\beta+\beta \Rightarrow \beta=\frac{x-y+z}{2}$$

άρα $\alpha=y-x+\dfrac{x-y+z}{2}=\dfrac{y-x+z}{2}$

Άρα $(x,y,z)=\dfrac{y-x+z}{2}(0,1,1)+\dfrac{x-y+z}{2}(1,0,1)+$

$+\dfrac{x+y-z}{2}(1,1,0)$

$$T(x,y,z) = \frac{y-x+z}{2} T(0,1,0) + \frac{x-y+z}{2} T(1,0,1) + \frac{x+y-z}{2}$$

$$T(1,1,0) =$$

$$= \frac{y-x+z}{2}(1,2) + \frac{x-y+z}{2}(2,1) + \frac{x+y-z}{2}(3,7) =$$

$$= \left(\frac{x-x+z+2x-2y+3x+3y-3z}{2}, \frac{2y-2y+2z+x-y+z-7x}{2} \right) =$$

$$= \left(\frac{4x+2y}{2}, \frac{6x+8y-4z}{2} \right) = (2x+y, 3x+4y-2z)$$

$$T(x,y,z) = (2x+y, 3x+4y-2z)$$

$\ker T$

$T(x,y,z) = (0,0)$

$2x+y=0 \Rightarrow y=-2x$

$3x+4y-2y=0 \Rightarrow 3x-8x-2z=0 \Rightarrow z=\frac{-5}{2}x$

$$\ker T = \{(x, -2x, -\tfrac{5}{2}x) \mid x \in \mathbb{R}\} = \langle (1,-2,-\tfrac{5}{2}) \rangle$$

$Y \leq \mathbb{R}^3$ ώστε $\ker Y \oplus Y = \mathbb{R}^3$

$\dim Y = 2$ αφού $\ker Y = 1$

$\Rightarrow Y = \langle (0,1,0), (0,0,1) \rangle = \langle (0,1,3), (0,1,1) \rangle$

$= \langle (0,2,-1), (0,1,1) \rangle$

Είναι ο υπόχωρος Y μοναδικός;

ΟΧΙ

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ΛΥΣΗ

α) $T(1,0,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$T(0,1,0) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

β) $B = (T, \underbrace{S \to S}_{\text{βάσεις}})$

$T(1,1,1) = (2,2,2) = 2(1,1,1) + 0(1,-1,0) + 0(1,1,-2)$

$T(1,-1,0) = (-1,0,-1) = \alpha(1,1,1) + \beta(1,-1,0) + \gamma(1,1,-2)$ (+)

$T(1,1,-2) = (-1,2,-1) = \delta(1,1,1) + \varepsilon(1,-1,0) + \zeta(1,1,-2)$ (++)

Από (+) $\left.\begin{aligned} -1 &= \alpha+\beta+\gamma \\ 0 &= \alpha-\beta+\gamma \end{aligned}\right\} \xrightarrow{\ominus} -1 = 2\beta \Leftrightarrow \beta = -\frac{1}{2}$

$-1 = \alpha - 2\gamma \Rightarrow \alpha = 2\gamma - 1$

$0 = 2\gamma - 1 + \frac{1}{2} + \gamma \Leftrightarrow 3\gamma = \frac{1}{2} \Leftrightarrow \gamma = \frac{1}{6}$

$\alpha = \frac{1}{3} - 1 = -\frac{2}{3}$

Άρα (+) $\to \frac{2}{3}(1,1,1) - \frac{1}{2}(1,-1,0) + \frac{1}{6}(1,1,-2)$

Από (++) $-1 = \delta+\varepsilon+\zeta \Rightarrow -3 = 2\varepsilon \Rightarrow \varepsilon = -\frac{3}{2}$

$2 = \delta-\varepsilon+\zeta \Rightarrow 2 = 2\zeta - 1 + \frac{3}{2} + 1 \Rightarrow 3\zeta = \frac{3}{2} \Rightarrow \zeta = \frac{1}{2}$

$-1 = \delta - 2\zeta \Rightarrow \delta = 2\zeta - 1 \Leftrightarrow \delta = 0$

Άρα (++) $\Rightarrow 0(1,1,1) - \frac{3}{2}(1,-1,0) + \frac{1}{2}(1,1,-2)$

Άρα $B = \begin{pmatrix} 2 & -\frac{2}{3} & 0 \\ 0 & -1/2 & -3/2 \\ 0 & 1/6 & 1/2 \end{pmatrix}$

$A = (T_1$ κανονικές βάσεις$)$

Πως σχετίζεται ο A με τον B; Αν ο P είναι ο πίνακας αλλαγής βάσης από την S στην κανονική τότε ο P^{-1} θα είναι πίνακας αλλαγής βάσης από την κανονική στην S'. Άρα

$$\begin{tikzcd} \text{καν} \arrow[r, "A"] \arrow[d, "P^{-1}"'] & \text{καν} \\ S' \arrow[r, "B"] & S \arrow[u, "P"'] \end{tikzcd}$$

$$\boxed{\begin{array}{c} A = PBP^{-1} \\ P^{-1}AP = B \end{array}}$$